

Math 250A Lecture 15 Notes

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1 Polynomials and Divisibility

1.1 Polynomial division with remainder

We start with some results you should already know.

Theorem 1.1. *Suppose f, g are polynomials in $R[x]$, where R is a commutative ring. Also suppose that f has leading coefficient 1, so $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$. Then $g(x) = f(x)q(x) + r(x)$, where $\deg(r) < \deg(f)$.*

Corollary 1.1. *If K is a field, $K[x]$ is a Euclidean domain.*

Proof. We can make the leading coefficient of any polynomial 1 by multiplying by a unit. Then apply the theorem. □

Corollary 1.2. *If K is a field, $K[x]$ is a principal ideal domain.*

Proof. All Euclidean domains are PIDs. □

Corollary 1.3. *If K is a field, $K[x]$ is a unique factorization domain.*

Proof. All PIDs are UFDs. □

Example 1.1. How can we find the prime elements of $F_2[x]$, where $F_2 = \mathbb{Z}/2\mathbb{Z}$, the field with 2 elements? Recall the sieve of Eratosthenes¹. List all numbers > 1 , identify the smallest number as prime, and cross out all multiples of it.

2, 3, ~~4~~, 5, ~~6~~, 7, ~~8~~, 9, ~~10~~, . . .

Then, identify the first non-crossed out number as prime, and cross out all multiples of it.

2, 3, ~~4~~, 5, ~~6~~, 7, ~~8~~, ~~9~~, ~~10~~, . . .

¹Eratosthenes was the first person to accurately calculate the circumference of the Earth.

If we repeat this process, we can find all the prime numbers.

For $F_2[x]$, we list all elements (other than 0 or units) in order of degree.

$$x, x + 1, x^2, x^2 + 1, x^2 + x, x^2 + x + 1, \dots$$

Cross out all multiples of x .

$$x, x + 1, \cancel{x^2}, x^2 + 1, \cancel{x^2 + x}, x^2 + x + 1, \dots$$

The next element, $x + 1$, is prime, so we cross out multiples of it. Note that $x^2 + 1 = (x + 1)^2$ in $F_2[x]$.

$$x, x + 1, \cancel{x^2}, \cancel{x^2 + 1}, \cancel{x^2 + x}, x^2 + x + 1, \dots$$

The polynomials not divisible by x and $x + 1$ are

$$x^2 + x + 1, x^3 + x + 1, x^3 + x^2 + 1, x^4 + x + 1, \cancel{x^4 + x^2 + 1}, x^4 + x^3 + 1, x^4 + x^3 + x^2 + x + 1,$$

and we can continue the process.

Proposition 1.1. *Suppose a polynomial $f \in R[x]$ has a root a ($f(a) = 0$). Then $f(x) = g(x)(x - a)$ for some g .*

Proof. Apply division to get that $f(x) = g(x)(x - a) + r$. We have $\deg(r) < 1$, so r is constant. Put $x = a$ to get $f(a) = g(a)(a - a) + r = r$, so $r = 0$. \square

Corollary 1.4. *A polynomial $f \in R[x]$ of degree n over an integral domain R has $\leq n$ roots.*

Proof. If a_1, \dots, a_k are roots, then $f(x) = (x - a_1) \cdots (x - a_k)g(x)$, so $k \leq n$. If the product is 0, then so is some factor $(x - a_i)$ because R is an integral domain. \square

Example 1.2. Let $R = \mathbb{Z}/8\mathbb{Z}$, which is not an integral domain. Let $f(x) = x^2 - 1$, which has degree 2. Then $f(x)$ has 4 roots: 1, 3, 5, and 7.

Example 1.3. Let R be the quaternions (this is noncommutative), and look at $f(x) = x^2 + 1$. Then f has roots $\pm i, \pm j, \pm k$, and roots $ai + bj + ck$ for real a, b, c that satisfy $a^2 + b^2 + c^2 = 1$. This is an uncountable number of roots!

1.2 An application to field theory

We first prove a lemma.

Lemma 1.1. *Any abelian group G with $\leq n$ elements of order n ($\forall n \geq 1$) is cyclic.*

Proof. Recall that $G \cong \mathbb{Z}/p_1^{n_1}\mathbb{Z} \times \mathbb{Z}/p_2^{n_2}\mathbb{Z} \times \cdots$. Suppose that $p_1 = p_2$; then G has p^2 elements x with $x^p = 1$ (since G contains $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$). This is impossible, so all p_i are distinct. Then G is cyclic by the Chinese remainder theorem ($\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/mn\mathbb{Z}$ if m, n are coprime). \square

Proposition 1.2. *The group $(\mathbb{Z}/p\mathbb{Z})^*$ of units mod p is cyclic if p is prime.*

Proof. Since p is prime, $\mathbb{Z}/p\mathbb{Z}$ is a field. So any polynomial in $R[x]$ of degree n has $\leq n$ roots. So $x^n - 1$ has $\leq n$ roots for any $n \geq 1$. Then G has $\leq n$ elements x with $x^n = 1$ (for $n \geq 1$). Using the lemma finishes off the proof. \square

Example 1.4. This need not hold if p is not prime. $(\mathbb{Z}/12\mathbb{Z})^* \cong (\mathbb{Z}/4\mathbb{Z})^* \times (\mathbb{Z}/3\mathbb{Z})^*$, and these are both cyclic of order 2.

Definition 1.1. A generator of $(\mathbb{Z}/p\mathbb{Z})^*$ is called a *primitive root*.

We have shown that primitive roots always exist when p is prime.

Example 1.5. Let's find a primitive root of $p = 23$. The element should have order 22. Check the elements $-1, 1, 2, 3, 4, 5$. We find that 5 is the primitive root because $5^2, 5^{11} \not\equiv 1 \pmod{23}$.

The same argument shows that the following is true.

Theorem 1.2. *If F is a field, any finite subgroup of F^* is cyclic.*

Example 1.6. Let $F = \mathbb{C}$, and take the subgroup of 8th roots of unity. This has primitive root $e^{2i\pi/8}$.

This also gives us the following corollary.

Corollary 1.5. *If F is any finite field, then F^* is cyclic.*

1.3 Unique factorization in polynomial rings

We want to show that $\mathbb{Z}[x]$ is a UFD, and we know that $\mathbb{Z}[x] \subseteq \mathbb{Q}[x]$, which is a UFD because \mathbb{Q} is a field. We cannot do this as we usually do, because $\mathbb{Z}[x]$ is not a Euclidean domain or a PID. For example, $(2, x)$ is a non-principal ideal. So we use the fact that $\mathbb{Q}[x]$ is a UFD.

Definition 1.2. Let $f \in \mathbb{Q}[x]$. The *content* $c(f)$ is defined as follows: Suppose $f(x) = a_n x^n + \cdots + a_0$. For each prime p , $a_n = p^{m_n} b_n, a_{n-1} = p^{m_{n-1}} b_{n-1}, \dots$ with $m_i \in \mathbb{Z}$ and b_i not having any factors of p in the numerator or denominator. Let $c(f) = p^{\min(m_i)} \times b$, where b is some number with no factors of p .

Example 1.7. Let $f(x) = (2/3)x^2 + 4$. Then $c(f) = 2/3$.

Proposition 1.3. $\mathbb{Z}[x]$ is a unique factorization domain.

Proof. The key point of the proof is that $c(fg) = c(f)c(g)$. We may assume that $c(f) = c(g) = 1$; otherwise, we multiply f and g by constants to make this so. We want to show that $c(fg) = 1$. We know that f has integer coefficients, so $c(f) \in \mathbb{Z}$. Suppose p is any prime in \mathbb{Z} ; we show that p does not divide $c(fg)$.

Since $c(f) = c(g) = 1$, p does not divide all coefficients of f or all the coefficients of g . So $f = a_nx^n + \cdots + a_ix^i + \cdots + a_0$ and $g = b_mx^m + \cdots + b_jx^j + \cdots + b_0$ where i and j are the least indices such that a_i and b_j are not divisible by p . So the coefficient of x^{i+j} in fg is

$$a_0b_{i+j} + a_1b_{i+j-1} + a_2b_{i+j-2} + \cdots + a_ib_j + \cdots + a_{i+j-1}b_1 + a_{i+j}b_0,$$

which has all terms except a_ib_j divisible by p . This means that the coefficient of x^{i+j} in fg is not divisible by p . This is true for any prime p , so $c(fg) = 1$.

We sketch the rest of the proof. The main point is that we need to show that irreducible elements are prime. Recall that irreducible elements are such that $f \neq gh$ with $\deg(g), \deg(h) < \deg(f)$; prime elements are such that if f divides g, h , then f divides g or h .

The irreducibles of $\mathbb{Z}[x]$ are the primes $2, 3, 5, 7, \dots \in \mathbb{Z}$ and the polynomials $f(x)$ of degree > 1 with $c(f) = 1$.

We leave the following two statements as exercises:

1. These are all the irreducibles of $\mathbb{Z}[x]$.
2. Any element of $\mathbb{Z}[x]$ is a product of irreducibles.

If $\deg(f) = 0$, then $f = p$ is prime in \mathbb{Z} . If f divides gh , this means that $c(gh)$ is divisible by p . So $c(g)$ or $c(h)$ is divisible by p (since $c(gh) = c(g)c(h)$). So p divides gh . The case of $\deg(f) > 0$ is similar and left as an exercise. \square

We have really proved the following theorem.

Theorem 1.3. *If R is a UFD, then so is $R[x]$.*

Proof. Perform the same proof but with a few modifications. First, $c(f)$ is now only defined up to multiplication by a unit. Also, irreducibles of $R[x]$ are either irreducibles of R ($\deg = 0$) or irreducibles of $K[x]$ with content 1, where K is the quotient field of R . \square

Corollary 1.6. $\mathbb{Z}[x_1, \dots, x_n]$ is a unique factorization domain.²

Corollary 1.7. *If K is a field, $K[x_1, \dots, x_n]$ is a unique factorization domain.*

Proof. These two have the same proof: induction on the number of variables. \square

²In fact, $\mathbb{Z}[x_1, x_2, \dots]$ in infinitely many variables is a field, but we will not prove that here.

1.4 Irreducibility tests in $\mathbb{Z}[x]$ (or $\mathbb{Q}[x]$)

Given $f \in \mathbb{Z}[x]$, how do we factor f into irreducibles?

Example 1.8. Here is an algorithm, due to Kronecker:

Suppose that $f = gh$. We can assume $g, h \in \mathbb{Z}[x]$. Then $f(n) = g(n)h(n)$ for any $n \in \mathbb{Z}$. So we factor $f(0), f(1), \dots, f(m)$, where $m = \deg(f)$. Then $g(0)$ divides $f(0)$ or $g(1)$ divides $f(1)$, (and so on), so there are only a finite number of possibilities for $g(0), \dots, g(m)$. But $\deg(g) \leq m$, so g is determined by $g(0), \dots, g(m)$.

Kronecker's algorithm is pretty slow. There are faster algorithms.

Example 1.9. The LLL algorithm³ is fast but not necessarily precise. We can write $f = af_1f_2 \cdots f_n$, where f_i is irreducible with degree > 0 and $a \in \mathbb{Z}$. We can do this in polynomial time, but to find a , we must factor an integer, which may not be possible in polynomial time.

To test for reducibility, we can use reduction mod p : If $f(x) = g(x)h(x)$, then $f(x) = g(x)h(x) \pmod{p}$ for any prime p .

Example 1.10. Is $9x^4 + 6x^3 + 26x^2 + 13x + 3$ irreducible? Yes. It is $x^4 + x + 1 \pmod{2}$, and we saw that this was irreducible $\pmod{2}$.

Example 1.11. Let's test if $x^4 - x^2 + 3x + 1$ is irreducible.

$$\pmod{2} : x^4 + x^2 + x + 1 = (x + 1)(x^3 + x^2 + 1),$$

which are both irreducible $\pmod{2}$.

$$\pmod{3} : x^4 - x^2 + 1 = (x^2 + 1)^2.$$

which is also irreducible $\pmod{3}$.

Combine these results. The first one says that the only possible factorization is a degree 1 polynomial times a degree 3 polynomial. The second says that the only possible factorization is into 2 degree 2 polynomials. So the polynomial must be irreducible.

Theorem 1.4 (Eisenstein). *Suppose $f(x)$ has the following properties:*

1. *The leading coefficient is 1.*
2. *All other coefficients are divisible by p .*
3. *The constant term is not divisible by p^2 .*

Then f is irreducible.

³This stands for Lenstra, Lenstra, and Lovasz.

We will not prove this right now. First, let's see some examples.

Example 1.12. The polynomial $x^5 - 4x + 2$ is irreducible by Eisenstein's criterion.

Example 1.13. Look at the p -th roots of 1. These are the roots of the polynomial $x^p - 1 = (x - 1)(x^{p-1} + x^{p-2} + \cdots + x + 1)$. We want to show that the latter term is irreducible by Eisenstein's criterion. We need a trick to make this work. Put $z = x - 1$. Then

$$\begin{aligned} x^{p-1} + \cdots + x + 1 &= \frac{x^p - 1}{x - 1} = \frac{(z + 1)^p - 1}{z} \\ &= \frac{(z^p + pz^{p-1} + \frac{p(p-1)}{2}z^{p-2} + \cdots + pz + 1) - 1}{z} \\ &= z^{p-1} + pz^{p-2} + \cdots + p, \end{aligned}$$

so Eisenstein applies, and $z^{p-1} + pz^{p-2} + \cdots + p$ is irreducible. So $x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible, as desired.

Why does this work? The prime p is *totally ramified* in $\mathbb{Z}[\zeta]$, where $\mathbb{Z}^p = 1$. We have that p factorizes in $\mathbb{Z}[\zeta]$ as $(1 - \zeta)^{p-1}u$, where u is a unit.